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Cosmology and fermion confinement in a scalar-field-generated domain wall brane in five dimensions

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Abstract: We consider a brane generated by a scalar field domain wall configuration in 4+1 dimensions, interpolating, in most cases, between two vacua of the field. We study the cosmology of such a system in the cases where the effective four-dimensional brane metric is de Sitter or anti de Sitter, including a discussion of the bulk coordinate singularities present in the de-Sitter case. We demonstrate that a scalar field kink configuration can support a brane with dS_4 cosmology, despite the presence of coordinate singularities in the metric. We examine the trapping of fermion fields on the domain wall for nontrivial brane cosmology.

Keywords: Solitons Monopoles and Instantons, Field Theories in Higher Dimensions, Large Extra Dimensions, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

Models incorporating more than $3+1$ dimensions have been of great interest to the high-energy physics community from the early 1980s onward, and have been investigated in considerable depth over the last decade (see, for instance, refs. [1 – 4]). It was initially thought that any such extra dimensions would need to be compactified to a small radius in order for their effect on four-dimensional gravity to be consistent with current experiments. However, the work of Randall and Sundrum [3, 4] showed that effective four-dimensional gravity could be recovered even in the case of non-compact extra dimensions.

The original Randall-Sundrum proposals, and much of the subsequent work, dealt with Standard Model fields confined a priori to a 3+1-dimensional subspace of the higher-dimensional space, with gravity confined to this "brane" by a warped five-dimensional metric (reviews are given in [5, 6]). In these models the brane is simply the boundary between two domains in the bulk spacetime, with the energy-momentum localised on the brane corresponding to a discontinuity in the derivatives of the metric across the brane. The ensuing cosmological evolution of the brane-bulk system has been studied extensively in the literature ([7, 8], or [8, 9] for discussions of cosmology induced by moving branes). Many modifications and extensions to the initial RS scheme have also been proposed: see, for example [8 – 13].

It has been suggested that such a 3+1-dimensional brane could also be realised as a smooth domain wall that dynamically confines the fields of the Standard Model [1, 14–19]. General aspects of smooth analogues of the Randall-Sundrum model have been studied in [20, 21]. In particular, given a scalar field potential with two neighbouring minima, a domain wall soliton may form where the scalar field interpolates between two regions of spacetime corresponding asymptotically to vacua of the field. This approach is appealing because it provides a dynamical origin for a brane model with one infinite extra dimension.

A number of authors have investigated solutions to the Einstein equations for a five-dimensional warped (nonfactorisable) metric coupled to a scalar field [14–16, 18, 19, 22–27] (see also [28, 29] for non-minimally coupled scalar-gravity models). The trapping of fermion fields on a subspace in models with extra dimensions has also been explored [1, 14, 17, 18, 30–35]. However, most (but not all) of these investigations have only considered the case where the effective 4D metric is Minkowskian, so they do not explore the cosmology of these models. The purpose of this paper is to study aspects of the cosmology of domain-wall-style branes.

In section 2 we write down the Einstein equations for a warped 5D metric based on an arbitrary 4D metric, with the energy-momentum tensor given by the bulk scalar field. We then discuss the properties of solutions to the equations of motion, in the cases where the effective 4D metric is Minkowski, dS_4 and AdS_4.

After reviewing some examples of scalar field domain walls with 4D Minkowski slices (section 3), we present criteria for a smooth 5D metric containing no curvature singularities with dS_4 slices in section 4. In section 5 we write down one such metric analytically, and plot the corresponding scalar field and potential for a particular parameter set. In section 6 we present new analytic solutions for the 5D warped metric with AdS_4 and dS_4 slices on the brane, and the associated background scalar field, motivated by thin-brane solutions that have been employed to demonstrate localisation of gravity [36].

Finally, in section 7 we consider the trapping of fermions in a spacetime described by a 5D warped metric (with dS_4 or AdS_4 brane cosmology) and supported by a kink-like scalar field domain wall. In section 8 we examine the behaviour of the chiral fermion zero modes in these various solution systems.

2. The equations of motion for a warped metric coupled to a scalar

Suppose $g^{(4)}_{\mu\nu}$ is some general 4D metric. Let us consider a 5D warped metric g_{MN} of the simple form,

$$g_{MN} = \begin{pmatrix} e^{-2\sigma(y)} g^{(4)}_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

(In general, we shall use capitalised Roman letters to indicate five-dimensional indices, and Greek letters for four-dimensional indices.) We use the sign conventions of [37]: in particular, the $5D$ metric has signature $(-++++)$.

This is a generalisation of the usual Randall-Sundrum warped metric ansatz [3, 4] to the case of a non-trivial four-dimensional metric. Such cosmological solutions have been studied in the context of infinitely thin branes by a number of authors [6–8, 12]. We shall refer to $e^{-\sigma(y)}$ as the "warp factor" associated with the warped metric, and $\sigma(y)$ as the "warp factor exponent".

This simple ansatz clearly does not encompass all possible 5D metrics that yield effective 4D gravity. In particular, if the bulk component T_{44} of the energy-momentum tensor depends only on the bulk coordinate, then the Einstein equations require that the effective 4D metric generated by this ansatz must be of constant curvature, allowing dS_4 and AdS_4 brane cosmology, but prohibiting more general FRW solutions.

It is well known that in the case of an infinitely thin brane, the presence of matter on the brane causes mixing of the bulk and brane coordinates in the metric, in the coordinate system where the brane is a hypersurface of constant y [8, 38]. This effect is frequently termed "brane bending". A simple concrete example is shown in [6], where an FRW ansatz is used for the 5D metric, the brane lies at $y = 0$, and the metric components are separable into bulk and brane coordinates only so long as the brane tension is the sole contributor to the energy-momentum tensor. However, the ansatz of eq. (2.1) is useful for studying systems where the brane tension dominates the matter contribution to the energy-momentum tensor, as in some cases the ensuing Einstein equations can be solved exactly.

Denoting the Ricci scalar computed from the 5D metric g_{MN} by $R^{(5)}$, and the 4D Ricci scalar computed from $g^{(4)}_{\mu\nu}$ by $R^{(4)}$, we find that the 5D Ricci scalar and Einstein tensor can be written in terms of their 4D counterparts and the warp factor as per

$$G_{MN} = \begin{pmatrix} G^{(4)}_{\mu\nu} + 3g^{(4)}_{\mu\nu} e^{-2\sigma(y)} \left(\sigma''(y) - 2\left(\sigma'(y)\right)^2\right) & 0 \\ 0 & -\frac{1}{2}e^{2\sigma(y)} R^{(4)} - 6\left(\sigma'(y)\right)^2 \end{pmatrix}, \tag{2.2}$$

$$R^{(5)} = e^{2\sigma(y)} R^{(4)} + \left(20\left(\sigma'(y)\right)^2 - 8\sigma''(y)\right). \tag{2.3}$$

The Einstein equations in five dimensions are,

$$G_{MN} = -8\pi G T_{MN}, \tag{2.4}$$

and in particular, writing $8\pi G = \kappa^2$, we see that

$$\kappa^2 T_{44} = \left(\frac{1}{2}e^{2\sigma(y)} R^{(4)} + 6\left(\sigma'(y)\right)^2\right), \tag{2.5}$$

implying that a time-varying 4D curvature scalar is associated with a time-dependent T_{44} (similarly, variation of the curvature scalar with the other brane coordinates requires T_{44} to depend on those coordinates). Thus if the brane tension is independent of the brane coordinates (t, x^1, x^2, x^3), and the trapped matter (if present) does not contribute to T_{44}, only constant-curvature solutions are possible with this metric ansatz.

Since we expect these four-dimensional metrics to be valid only in the case where the brane tension dominates over the matter contribution to the energy-momentum tensor, let us compute the 5D energy-momentum tensor derived from a scalar field Φ and associated potential $V(\Phi)$.

In the absence of matter sources, the system is described by the usual Einstein-Hilbert action with additional scalar field terms [37, 39],

$$
\begin{aligned}
S &= \int d^5x \sqrt{-g(x)} \left[\frac{-1}{2\kappa^2} R^{(5)}(x) + \mathcal{L}_\Phi \right] && (2.6)\\
\mathcal{L}_\Phi &= -\frac{1}{2} g^{MN}\, \partial_M \Phi\, \partial_N \Phi \,-\, V(\Phi), && (2.7)\\
g(x) &\equiv \mathrm{Det}\,(g_{MN})\,. && (2.8)
\end{aligned}
$$

The resulting energy-momentum tensor is easily computed as,

$$
\begin{aligned}
T^{MN} &= 2\,\frac{\delta \mathcal{L}_\Phi}{\delta g_{MN}} + g^{MN} \mathcal{L}_\Phi \\
&= g^{MS} g^{NR} \partial_R \Phi\, \partial_S \Phi - g^{MN} \left(V(\Phi) + \frac{1}{2} g^{RS}\, \partial_R \Phi\, \partial_S \Phi \right). \qquad (2.9)
\end{aligned}
$$

Suppose the scalar field Φ is a function of the bulk coordinate y only. Then using the separable metric ansatz of eq. (2.1), the non-zero elements of the energy-momentum tensor become,

$$
\begin{aligned}
T_{\mu\nu} &= -e^{-2\sigma(y)} g^{(4)}_{\mu\nu} \left(V(\Phi) + \frac{1}{2} \left(\Phi'(y)\right)^2 \right), && (2.10)\\
T_{44} &= \frac{1}{2} \left(\Phi'(y)\right)^2 - V(\Phi). && (2.11)
\end{aligned}
$$

The Einstein equations, $G_{MN} = -\kappa^2 T_{MN}$, and eqs. (2.2)-(2.3), then yield:

$$
\begin{aligned}
G^{(4)}_{\mu\nu} &= e^{-2\sigma(y)} g^{(4)}_{\mu\nu} \left[\kappa^2 \left(V(\Phi) + \frac{1}{2} \left(\Phi'(y)\right)^2 \right) - 3 \left(\sigma''(y) - 2 \left(\sigma'(y)\right)^2 \right) \right], && (2.12)\\
R^{(4)} &= 2 e^{-2\sigma(y)} \left[-6 \left(\sigma'(y)\right)^2 + \kappa^2 \left(\frac{1}{2} \left(\Phi'(y)\right)^2 - V(\Phi) \right) \right]. && (2.13)
\end{aligned}
$$

For these equations to be consistent, there must be some constant γ such that $G^{(4)}_{\mu\nu} = \gamma g^{(4)}_{\mu\nu}$ ($\Rightarrow R^{(4)} = -4\gamma$). Then we obtain the equations of motion,

$$
\begin{aligned}
\gamma e^{2\sigma(y)} &= \kappa^2 \left(V(\Phi) + \frac{1}{2} \left(\Phi'(y)\right)^2 \right) - 3 \left(\sigma''(y) - 2 \left(\sigma'(y)\right)^2 \right), && (2.14)\\
&= 3 \left(\sigma'(y)\right)^2 - \frac{\kappa^2}{2} \left(\frac{1}{2} \left(\Phi'(y)\right)^2 - V(\Phi) \right). && (2.15)
\end{aligned}
$$

We can eliminate the potential from these equations and obtain the result,

$$
3\sigma''(y) - \kappa^2 \left(\Phi'(y)\right)^2 = \gamma e^{2\sigma(y)}. \qquad (2.16)
$$

The potential V can easily be obtained as a function of y from either of eqs. (2.14)-(2.15),

$$
V(y) = \frac{1}{2} \left(\Phi'(y)\right)^2 - \frac{6}{\kappa^2} \left(\sigma'(y)\right)^2 + \frac{2\gamma}{\kappa^2} e^{2\sigma(y)}. \qquad (2.17)
$$

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In the case of AdS_4 space, γ is negative, while for dS_4 space the converse holds true. Minkowski space corresponds to the case $\gamma = 0$.

For $\gamma \neq 0$, we can define the dimensionless quantities,

$$\eta = y\sqrt{\frac{|\gamma|}{3}}, \tag{2.18}$$

$$\hat{\Phi}(\eta) = \frac{\kappa}{\sqrt{3}}\Phi\left(y\sqrt{\frac{|\gamma|}{3}}\right), \tag{2.19}$$

$$\hat{V}(\eta) = \frac{\kappa^2}{|\gamma|}V\left(y\sqrt{\frac{|\gamma|}{3}}\right). \tag{2.20}$$

In terms of these dimensionless quantities, eqs. (2.16) and (2.17) become,

$$e^{2\sigma(\eta)} = \frac{|\gamma|}{\gamma}\left(\sigma''(\eta) - \left(\hat{\Phi}'(\eta)\right)^2\right), \tag{2.21}$$

$$\hat{V}(\eta) = \frac{1}{2}\left(\hat{\Phi}'(\eta)\right)^2 - 2\left(\sigma'(\eta)\right)^2 + 2\frac{\gamma}{|\gamma|}e^{2\sigma(\eta)}. \tag{2.22}$$

Note that, independent of the value of γ, differentiating eq. (2.15) and then substituting Equation (2.16) yields the Klein-Gordon equation for the scalar field (this relationship between the Einstein equations and the Klein-Gordon equation is discussed in [20]),

$$\frac{\mathrm{d}V}{\mathrm{d}\Phi} = \Phi''(y) - 4\sigma'(y)\Phi'(y), \tag{2.23}$$

or using the dimensionless quantities defined above,

$$\frac{\mathrm{d}\hat{V}}{\mathrm{d}\hat{\Phi}} = \hat{\Phi}''(\eta) - 4\sigma'(\eta)\hat{\Phi}'(\eta). \tag{2.24}$$

Later we will wish to discuss configurations where the metric elements go to zero at some point in the bulk. In this case, σ is an unsuitable parameterisation as it diverges to ∞ at zeroes of the warp factor. If in eq. (2.1) we replace $e^{-2\sigma}$ with ω^2, where ω is a real function of the bulk coordinate, then the equations of motion (eqs. (2.16)-(2.17)) can be computed exactly as previously, and become,

$$3\left(\omega'(y)\right)^2 - 3\omega(y)\omega''(y) - \kappa^2\left(\Phi'(y)\right)^2\omega(y)^2 = \gamma, \tag{2.25}$$

$$V(y) = \frac{1}{2}\left(\Phi'(y)\right)^2 - \frac{6}{\kappa^2}\left(\frac{\omega'(y)}{\omega(y)}\right)^2 + \frac{2\gamma}{\kappa^2\omega(y)^2}. \tag{2.26}$$

(Note: It immediately follows from these equations of motion that if the first and second derivatives of ω with respect to the bulk coordinate remain finite, then ω may possess zeroes only if $\gamma \geq 0$, i.e. where the 4D metric is dS_4 or Minkowski. We will later show that zeroes in ω inevitably arise for the case of a smooth ω with dS_4 brane cosmology.)

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In terms of the dimensionless quantities defined in eqs. (2.18)-(2.20), the equations of motion for ω are,

$$\frac{|\gamma|}{\gamma} = \left(\omega'(\eta)\right)^2 - \omega''(\eta)\omega(\eta) - \omega(\eta)^2\left(\hat{\Phi}'(\eta)\right)^2, \tag{2.27}$$

$$\hat{V}(\eta) = \frac{1}{2}\left(\hat{\Phi}'(\eta)\right)^2 - 2\left(\frac{\omega'(\eta)}{\omega(\eta)}\right)^2 + 2\frac{\gamma}{|\gamma|}\frac{1}{\omega(\eta)^2}. \tag{2.28}$$

We will preferentially use the σ notation when not dealing directly with cases where $\sigma \to \infty$, for ease of comparison with the literature. Coordinate singularities in the metric, where σ diverges (and ω goes to zero), correspond to bulk horizons, and most previous braneworld studies have been concerned only with the behaviour of the brane-bulk system in the region bounded by these horizons.

3. Scalar field kink configurations supporting a Minkowski brane

In the case where $\gamma = 0$ and the brane metric corresponds to Minkowski space, eq. (2.16) can be integrated to give a warp factor corresponding to any chosen scalar field profile. The potential generating this scalar field can then be analytically determined from eq. (2.17) with $\gamma = 0$. Many solutions to the coupled gravity-scalar system for a Minkowski brane have been discussed in the literature (see for example [16, 18, 21, 26, 30, 31]).

Two simple examples are the kink configurations $\Phi_1(y) = (\sqrt{3}A/\kappa)\tanh(ry)$, $\Phi_2(y) = (\sqrt{3}A/\kappa)\arctan\sinh(ry)$, where A is a dimensionless constant and r has dimensions of inverse length. Eq. (2.16) then yields the corresponding warp factor exponents σ_1, σ_2:

$$\sigma_1(y) = A^2\left(\frac{2}{3}\ln\cosh(ry) - \frac{1}{6}\mathrm{sech}^2(ry) + Bry + C\right), \tag{3.1}$$

$$\sigma_2(y) = A^2\left(\ln\cosh(ry) + Bry + C\right), \tag{3.2}$$

where B and C are dimensionless constants of integration. If the warp factor exponent is an even function of y (a usual symmetry of domain-wall solutions and the RS model with a single brane [4]), then it follows that the integration constant B must be zero. Then in both cases, the warp factors $e^{-\sigma}$ converge to zero far from the brane, like the warp factor of [4] for the case of a fine-tuned infinitely thin brane. The corresponding potentials V_1, V_2 can easily be obtained from eq. (2.17): V_1 is a sextic polynomial in Φ [18], while $V_2(\Phi)$ is

$$V_2(\Phi) = -\frac{3A^2}{2}\frac{r^2}{\kappa^2}\left[\left(4A^2+1\right)\sin^2\left(\frac{\kappa}{A\sqrt{3}}\Phi\right) + 8A^2B\sin\left(\frac{\kappa}{A\sqrt{3}}\Phi\right) + \right. \\ \left. + \left(4A^2B^2 - 1\right)\right]. \tag{3.3}$$

If B vanishes (required if the warp factor is even) then the potential is also an even function of Φ.

This method is useful for deriving analytically tractable solutions, but is conceptually somewhat unsatisfactory because the system is fundamentally determined by the potential, not the profile of the scalar field. "Superpotential" methods have been used to reduce

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the equations of motion to a system of first-order equations by introducing an auxiliary potential [19, 23, 24], allowing the scalar field profile and metric corresponding to a wide range of potentials to be derived. These calculations can often be performed analytically, although writing the potential as a function of the scalar field rather than the bulk coordinate requires an inversion of the scalar field profile which is often analytically intractable.

If γ is non-zero, i.e. the 4D metric has non-zero constant curvature, then the problem becomes significantly more complex and this superpotential approach no longer yields easily solvable equations of motion. (This problem has been investigated in [19, 23, 24]).

A modified superpotential method can be employed to generate scalar field configurations and metric warp factors corresponding to a given auxiliary potential [23]. However, it is applicable only to a limited class of potentials. The solutions found by this method with AdS_4 brane cosmology exhibit kink-like scalar field configurations, but the dS_4 solutions contain naked curvature singularities in the metric, associated with infinities in the scalar field as the potential becomes unbounded below, at a finite distance from the brane.

4. Properties of solutions to the Einstein equations with dS_4 cosmology

Putting the superpotential approach aside, let us consider the general properties of solutions to the equations of motion (eqs. (2.21)-(2.22)) when $\gamma > 0$. We will prove that in this case, corresponding to dS_4 brane cosmology, there are no smooth, even solutions for the exponent $\sigma(\eta)$: if the warp factor $e^{-\sigma(\eta)}$ is to be smooth and even, it must possess zeroes.

We begin by noting that for $\gamma > 0$, $\sigma''(\eta) = (\hat{\Phi}'(\eta))^2 + e^{2\sigma(\eta)} > 0\,\forall\,\eta$. Thus $\sigma'(\eta)$ is monotonically increasing $\forall\,\eta$. Suppose that $\sigma(\eta)$ is an even continuous function of η with continuous and well-defined first and second derivatives: then $\sigma'(0) = 0$ and consequently $\sigma'(\eta)$ is negative for $\eta < 0$ and positive for $\eta > 0$. Thus if $\sigma(\eta)$ is a smooth function of η, then it must be strictly decreasing for $\eta < 0$ and strictly increasing for $\eta > 0$, and it follows that $\sigma(0)$ is a minimum of the metric exponent σ: $\sigma(\eta) > \sigma(0)\,\forall\,\eta \neq 0$.

Substituting back into eq. (2.21), it follows that

$$\sigma''(\eta) \geq e^{2\sigma(\eta)} \geq e^{2\sigma(0)}, \tag{4.1}$$

and integrating this inequality implies that $\sigma(\eta), |\sigma'(\eta)| \to \infty$ as $\eta \to \pm\infty$.

Let $g(\eta) \equiv \sigma'(\eta)e^{-\sigma(\eta)}$. Then we can rewrite eq. (2.21) in the form,

$$g'(\eta)e^{-\sigma(\eta)} + g(\eta)^2 = 1 + e^{-2\sigma(\eta)}\left(\hat{\Phi}'(\eta)\right)^2. \tag{4.2}$$

By hypothesis, $g(\eta)$ is a smooth odd function of η, with $g(\eta) > 0$ for $\eta > 0$, and $g'(0) = \sigma''(0)e^{-\sigma(0)} > 0$.

Now wherever $g'(\eta) \leq 0$, we must have $g(\eta)^2 \geq 1$, which for $\eta > 0$ implies $g(\eta) \geq 1$. Taking the contrapositive, in the range $\eta > 0$, whenever $g(\eta) < 1$, $\Rightarrow g'(\eta) > 0$. Thus if $g(\eta) < 1$, $\forall\,\eta > 0$, then g is bounded and monotonically increasing and converges to some (strictly) positive real number $\delta \leq 1$. Furthermore, since $g(\eta)$ is monotonically increasing on any interval for which $g(\eta) < 1$, and g is continuous, it follows that if $\exists\,\eta_0$ s.t. $g(\eta_0) \geq 1$, then $\eta \geq \eta_0 \Rightarrow g(\eta) \geq 1$.

In both of these cases, $\exists$ some real and strictly positive number ϵ, and some $\eta_0 > 0$, such that $\eta \geq \eta_0 \Rightarrow g(\eta) \geq \epsilon$. (In the first case, we can simply choose $\epsilon = \delta/2$.)

But then choosing some $\eta_1 > \eta_0$, and integrating $g(\eta)$ over this interval, we find,

$$\epsilon(\eta_1 - \eta_0) \leq \int_{\eta_0}^{\eta_1} g(\eta)\,\mathrm{d}\eta = \int_{\eta_0}^{\eta_1} \sigma'(\eta) e^{-\sigma(\eta)}\,\mathrm{d}\eta = -e^{-\sigma(\eta_1)} + e^{-\sigma(\eta_0)} \leq e^{-\sigma(\eta_0)} \leq e^{-\sigma(0)}. \tag{4.3}$$

But this is a contradiction, as by choosing η_1 arbitrarily large we can obtain an arbitrarily large lower bound on $e^{-\sigma(0)}$, which must be finite.

Consequently, there is no even, smooth warp factor exponent $\sigma(y)$ defined on the whole real line that yields a four-dimensional deSitter cosmology. It is possible that the warp factor $\omega(y)$ is smooth, but in this case it must contain zeroes (as smoothness of $\omega = \pm e^{-\sigma}$ implies smoothness of σ except where ω changes sign).

It is well known that warped metrics with dS_4 slices often contain horizons or naked singularities at a finite distance from the brane [12, 19, 25, 26, 36]. The properties of horizons in the case of a thin brane with no scalar field have been discussed in [12, 13], and curvature singularities in the presence of a scalar field were considered in [24, 25]. We have demonstrated the strong result that such features are inevitable in the case of a brane supported by a scalar field.

In the simplest analytic solutions for this system, the scalar field diverges to infinity at these points, and naked curvature singularities occur [23 − 25]. We will show that it is possible to avoid a diverging curvature at these points, and to obtain a kink-like configuration for the scalar field supporting the dS_4 brane, but at present we have only obtained a partially analytic example of such a solution.

The physical interpretation of singularities in the metric, for a model where gravity is coupled to a scalar field, has been discussed in the literature [25, 26]. In particular, Gremm suggests that five-dimensional spacetimes consisting of domain walls interpolating between spaces with naked curvature singularities may be interpreted as four-dimensional gravity coupled to a non-conformal field theory [25]. Gremm also claims to present a warped metric with dS_4 slices where the curvature remains finite at the metric zeroes, but the scalar field and potential diverge, but we have not been able to verify this solution. Davidson and Mannheim give a similar solution in the case of an infinitesimally thin brane, with a divergent scalar field and potential but bounded curvature at the metric zeroes [26], and suggest that such a solution might provide a mechanism for dynamical compactification of the extra dimension.

To distinguish between curvature and coordinate singularities, we examine the five-dimensional curvature scalar, given by eq. (2.3). Also, $-\kappa^2 T^M_M = G^M_M = R^M_M - (1/2) g^M_M R = R(1 - n/2)$, where n is the number of dimensions. So the five-dimensional curvature scalar can be expressed in terms of the trace of the energy-momentum tensor,

$$R^{(5)} = \frac{2\kappa^2}{3} T^M_M. \tag{4.4}$$

In the case where the energy-momentum tensor is generated entirely by the scalar field and

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potential,

$$
\begin{aligned}
T_M^M &= -4\left(V(\Phi) + \frac{1}{2}\left(\Phi'(y)\right)^2\right) + \frac{1}{2}\left(\Phi'(y)\right)^2 - V(\Phi) \\
&= -5V(\Phi) - \frac{3}{2}\left(\Phi'(y)\right)^2 .
\end{aligned} \tag{4.5}
$$

Thus, the curvature scalar can only become singular if the potential, the scalar field derivative, or both, diverge to infinity. If the potential and scalar field remain bounded at a zero in the metric, this is sufficient (albeit not necessary, as diverging terms on the r.h.s. of eq. (4.5) may cancel each other out) to ensure that the zero is not associated with a curvature singularity.

In the case where $\gamma \neq 0$, we can also write,

$$
R^{(5)} = e^{2\sigma(\eta)} R^{(4)} + \frac{|\gamma|}{3}\left(20\left(\sigma'(\eta)\right)^2 - 8\sigma''(\eta)\right) \tag{4.6}
$$

$$
= -|\gamma|\left(\frac{10}{3}\hat{V}(\hat{\Phi}(\eta)) + \left(\hat{\Phi}'(\eta)\right)^2\right). \tag{4.7}
$$

5. Partially analytic solution with dS_4 slices and no curvature singularities

Having demonstrated above that the dS_4 case must produce metric zeroes (or a divergent metric) in the bulk, to proceed with our analysis of that case we now no longer take the warp factor to be the exponential of a real-valued function. Instead, we work directly with the warp factor ω introduced previously, employing eqs. (2.27)-(2.28) as the equations of motion.

Let us denote the roots of $\omega(\eta)$ by η_0, i.e. $\omega(\eta_0) = 0$. Then in the dS_4 case where γ is positive, it follows immediately from eq. (2.27) that

$$
\omega'(\eta_0) = \pm 1, \tag{5.1}
$$

for the case of interest where ω'' and $\hat{\Phi}'$ remain finite at the metric zeroes. Differentiating both sides of the equation of motion up to fourth order and evaluating at $y = y_0$, we obtain the relations:

$$
\omega''(\eta_0) = 0, \tag{5.2}
$$

$$
\hat{\Phi}'(\eta_0) = 0, \tag{5.3}
$$

$$
\omega^{(4)}(\eta_0) = 0. \tag{5.4}
$$

(See also [24] for an alternate approach yielding eq. (5.3)).

If $\hat{\Phi}(\eta)$ is not $1:1$, then it cannot be inverted to yield an expression for $\hat{V}$ as a function of $\hat{\Phi}$, and thus $\hat{V}$ may fail to be a single-valued function of $\hat{\Phi}$. It is still *possible* that $\hat{V}$ might be a single-valued function of $\hat{\Phi}$ in any case, but we can ensure this is true by requiring $\hat{\Phi}$ to be nondecreasing. With this additional constraint, any point where $\hat{\Phi}'(\eta) = 0$ must also be a turning point in the derivative, i.e. $\hat{\Phi}''(\eta)$ must also be zero.

This assumption has consequences for the form of the required potential. We can rewrite the Klein-Gordon equation (eq. (2.24)) in terms of $\omega(\eta)$,

$$\frac{\mathrm{d}\hat{V}}{\mathrm{d}\hat{\Phi}} = \hat{\Phi}''(\eta) + 4\frac{\omega'(\eta)}{\omega(\eta)}\hat{\Phi}'(\eta). \tag{5.5}$$

If $\hat{\Phi}(\eta)$ is nondecreasing and continuous, then $\hat{\Phi}'(\eta) \geq 0 \,\forall\, \eta$, and as zeroes in the metric correspond to points where $\omega'(\eta)/\omega(\eta)$ changes sign, it follows that zeroes in the metric are either zeroes in the derivative of the potential with respect to $\hat{\Phi}$ (if $\mathrm{d}\hat{V}/\mathrm{d}\hat{\Phi}$ is continuous at these points), or cusps where $\mathrm{d}\hat{V}/\mathrm{d}\hat{\Phi}$ changes sign. This latter case was proposed as a possibility by Davidson and Mannheim [26].

Imposing the condition that $\hat{\Phi}''(\eta_0) = 0$ and differentiating the equation of motion up to sixth order yields the further relations:

$$\left(\omega^{(3)}(\eta_0)\right)^2 = \omega^{(1)}(\eta_0)\omega^{(5)}(\eta_0), \tag{5.6}$$

$$\omega^{(6)}(\eta_0) = 0\,. \tag{5.7}$$

Note that the conditions on the metric are a result of our somewhat artificial approach in starting from the metric and attempting to reconstruct a physically reasonable scalar field kink and potential. It is not clear to what degree they indicate constraints on the class of potentials that can give rise to a scalar kink supporting a dS_4 domain wall. However, we have also derived constraints that apply directly to the potential, implying that it must have turning points or points of inflection at the zeroes of the metric. This constitutes a limitation on the physical potentials that can give rise to a smooth singularity-free scalar field and metric with dS_4 cosmology on the brane. In flat space we would expect $\hat{V}(\hat{\Phi})$ to possess minima at the asymptotic values of the scalar field $\hat{\Phi}(\eta)$, that is at the limits $\lim_{\eta\to\pm\infty}\hat{\Phi}(\eta)$, but, as noted above, in the present case the potential must have extrema at the zeroes of the metric. The values of $\hat{\Phi}(\eta)$ at the zeroes of the metric will generally not coincide with the asymptotes, so a simple double-well potential cannot give rise to the desired metric and scalar field configuration.

A straightforward but inelegant method for writing down a warp factor satisfying these conditions is to start with a simple cosh function, motivated by the warp factor in the case of an infinitely thin brane with a time-dependent metric [6, 7, 36]. Adding additional linearly independent even functions (sech, sech^2 and sech^4 functions are suitable) which decrease rapidly far from the brane and are zero at the roots of the warp factor, we retain the essential features of the delta-function brane solution, but the coefficients of these terms can be adjusted to fine-tune the derivatives of the metric as required.

Once the metric has been written down, eq. (2.27) immediately yields the derivative of the dimensionless scalar field $\hat{\Phi}'(\eta)$, and integrating gives $\hat{\Phi}(\eta)$ itself. Equation (2.28) then yields the dimensionless potential $\hat{V}$ as a function of η, and inverting the function $\hat{\Phi}(\eta)$ yields an expression for $\hat{V}$ as a function of $\hat{\Phi}$.

However, at this stage the integration to obtain $\hat{\Phi}(\eta)$ (and consequently the inversion of this function to obtain $\hat{V}(\hat{\Phi})$) has not been performed analytically, due to the inelegance of our method for writing down a metric satisfying all the requirements of the problem.

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Specifically, the metric ansatz,

$$\omega(\eta) = A - R\cosh(r\eta) + B\left(\operatorname{sech}(r\eta) - \frac{R}{A}\right) + C\left(\operatorname{sech}^2(r\eta) - \left(\frac{R}{A}\right)^2\right) + \\ +D\left(\operatorname{sech}^4(r\eta) - \left(\frac{R}{A}\right)^4\right) + E\left(\operatorname{sech}^6(r\eta) - \left(\frac{R}{A}\right)^6\right), \tag{5.8}$$

satisfies eqs. (5.1)-(5.2) and (5.4)-(5.6), with A and R as adjustable parameters and B, C and D given by,

$$B = \{A^9\left(16A^{10} - 200R^2A^8 + 1294R^4A^6 - 3327R^6A^4 + 3612R^8A^2 - 1400R^{10}\right) + \\ +ER^6\left(-8192A^{12} + 73728A^{10}R^2 - 279040A^8R^4 + 567296A^6R^6 - \\ -649152A^4R^8 + +393792A^2R^{10} - 98448R^{12}\right)\} / \left[A^5R\left(16A^{12} - 168A^{10}R^2 + \\ +766A^8R^4 - 2019A^6R^6 + 3070A^4R^8 - 2432A^2R^{10} + 768R^{12}\right)\right], \tag{5.9}$$

$$C = \frac{1}{2}\{A^9\left(28A^8 - 443R^2A^6 + 1476R^4A^4 - 1840R^6A^2 + 784R^8\right) + ER^4\left(5600A^{12} - \\ -53648A^{10}R^2 + 217828A^8R^4 - 479506A^6R^6 + 595020A^4R^8 - 389424A^2R^{10} + \\ +104160R^{12}\right)\} / \left[A^4\left(16A^{12} - 168A^{10}R^2 + 766A^8R^4 - 2019A^6R^6 + 3070A^4R^8 - \\ -2432A^2R^{10} + 768R^{12}\right)\right], \tag{5.10}$$

$$D = \frac{1}{4}\{A^9\left(2A^8 + 13R^2A^6 - 80R^4A^4 + 128R^6A^2 - 64R^8\right) + ER^4\left(-896A^{12} + \\ +9056A^{10}R^2 - 39864A^8R^4 + 98164A^6R^6 - 137328A^4R^8 + 100416A^2R^{10} - \\ -29568R^{12}\right)\} / \left[R^2A^2\left(16A^{12} - 168A^{10}R^2 + 766A^8R^4 - 2019A^6R^6 + 3070A^4R^8 - \\ -2432A^2R^{10} + 768R^{12}\right)\right]. \tag{5.11}$$

The parameter E is obtained by solving the quadratic equation,

$$0 = E^2 \times (9408R^{12}A^2 - 11424R^{14}) + \\ +E \times (-1225R^4A^{11} + 5116BR^7A^7 - 2048CR^6A^8 - 8112CR^{10}A^4 + 5488A^9R^6 - \\ -5040A^7R^8 - 1225BR^5A^9 + 12832DR^{10}A^4 - 1600DR^8A^6 - 4488BR^9A^5 - \\ -13440DR^{12}A^2 + 9056CR^8A^6) + \\ +1600D^2R^8A^6 + 8B^2R^2A^{12} - 3BRA^{13}C + 72A^{13}CR^2 + 1000A^{11}DR^4 - \\ -20A^{12}BR^3 - 1120A^9DR^6 - 3CA^{15} - 150DR^2A^{13} - 120A^{11}CR^4 + \\ +76BR^3A^{11}C - 2080D^2R^{10}A^4 - 144C^2R^6A^8 + 8A^{14}BR - 960BR^7A^7D - \\ -112BR^5A^9C + 912BR^5A^9D - 150BR^3A^{11}D + 1568CR^6A^8D + 96C^2R^4A^{10} - \\ -1760CR^8A^6D - 192CR^4A^{10}D - 14B^2R^4A^{10}. \tag{5.12}$$

The parameter r is then determined by solving the equation,

$$1 = \left(\omega'(\eta_0)\right)^2 = \frac{r^2(A^2-R^2)(A^7 + BRA^5 + 2CR^2A^4 + 4DR^4A^2 + 6ER^6)^2}{A^{14}}. \tag{5.13}$$

For example, setting $A = 5$, $R = 1$, we obtain,

$$B \approx 24.0, C \approx 2.53, D \approx 8.17, E \approx -1.66, r \approx 0.101. \tag{5.14}$$

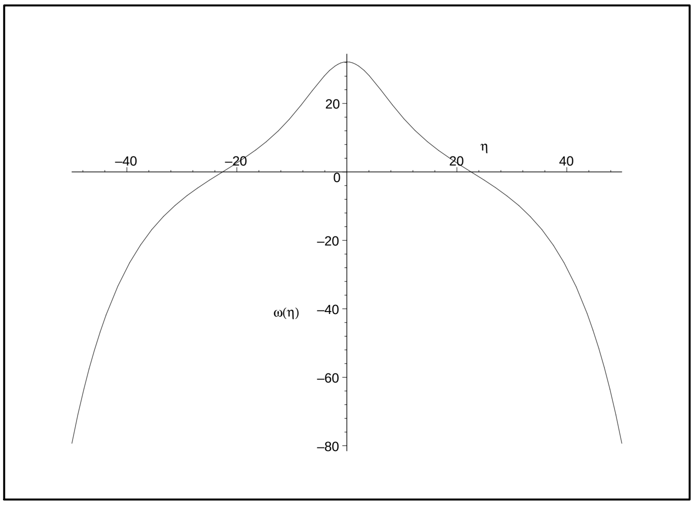

Figure 1: Warp factor $\omega(\eta)$ for the smooth kink with dS_4 brane cosmology; $A = 5, R = 1$, see eqs. (5.8)-(5.14).

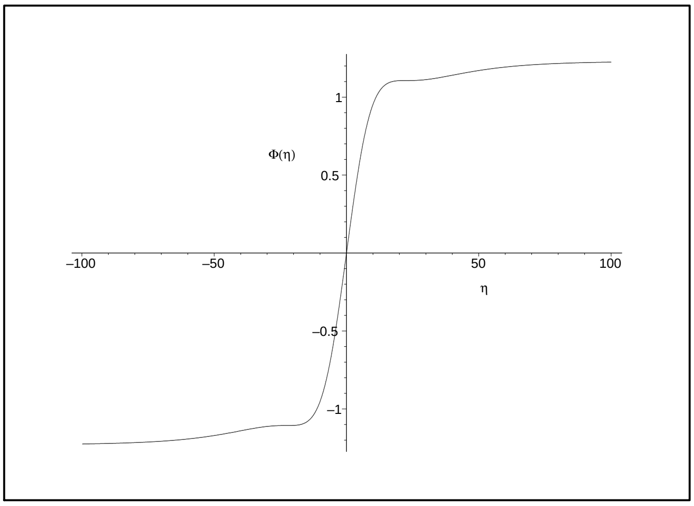

Figure 2: Dimensionless scalar field profile $\hat{\Phi}(\eta)$ generating dS_4 brane cosmology; $A = 5, R = 1$, see eqs. (5.8)-(5.14).

figures (1) and (2) illustrate the bulk profile of the metric and dimensionless scalar field in this case, while figure 3 depicts the required potential.

6. "Locally localised gravity" solutions with AdS_4 and dS_4 brane cosmology

In the case where $\gamma < 0$ and the brane cosmology is anti-deSitter, the metric zeroes which complicate the dS_4 case do not occur and it is possible to write down a fully analytic solution system. The simplest such solution (which may also be obtained by a superpotential approach [23] starting with a trigonometric periodic potential) has the warp factor exponent $\sigma(\eta) = A - \ln\cosh(r\eta)$. However, we shall first consider a slightly more complicated trial

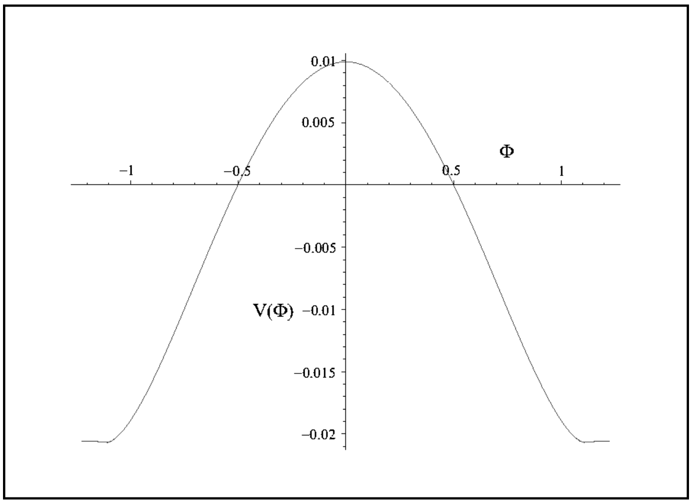

Figure 3: Potential $V(\Phi)$ generating dS_4 brane cosmology; $A = 5, R = 1$, see eqs. (5.8)-(5.14).

warp factor, and recover this simple solution as a special case.

Karch and Randall [36] discussed warp factors giving rise to localised four-dimensional gravity in the case of an infinitely thin brane. In the case of a brane with AdS_4 cosmology, the warp factor grew exponentially far from the brane, like the simple solution described above, or like the warped metric presented in [31] generated by a "ghost" scalar field. However, close to the brane the metric behaved qualitatively like the decreasing warped metric associated with a Minkowski brane (section 3, [4]), and it was this local behaviour that was responsible for confining gravity to the brane. It seems reasonable, then, that a similar but smooth metric (with a decreasing warp factor close to the brane that then turns around and increases exponentially) might be required to localise four-dimensional gravity on a scalar field domain wall in the AdS_4 case. An example of such a warp factor exponent is plotted in figure 4, and graviton confinement in coupled metric-scalar systems of this form was studied by Kobayashi et al [22].

It is possible to "smooth out" warp factors initially derived for a thin-brane system, essentially by replacing $|\eta|$ with $\ln(2\cosh\eta)$. Using this idea as motivation, we can construct exact analytic solutions for the scalar field and metric with AdS_4 and dS_4 slices: the supporting scalar field is kink-like in the AdS_4 case but contains curvature singularities in the dS_4 case, as discussed above.

We consider a trial warp factor of the form,

$$\omega(\eta) = A\cosh(r\eta) + B\mathrm{sech}(r\eta). \tag{6.1}$$

If $0 < A/B < 1$, then this metric has off-brane turning points similar to those in the solutions of [36, 22] (see figure 4). The limit where $B = 0$ corresponds to the simple solution mentioned at the start of this section. In any case, eq. (2.27) becomes:

$$\left(\hat{\Phi}'(\eta)\right)^2 = \frac{1}{\omega(\eta)^2}\left(\left(\omega'(\eta)\right)^2 - \omega''(\eta)\omega(\eta) - \frac{\gamma}{|\gamma|}\right)$$

$$= \frac{1}{(A\cosh(r\eta) + B\mathrm{sech}(r\eta))^2} \times \tag{6.2}$$

$$\times \left[-\frac{\gamma}{|\gamma|} + r^2 \left(-A(A+4B) + 4AB\mathrm{sech}^2(r\eta) + B^2\mathrm{sech}^4(r\eta) \right) \right]. \tag{6.3}$$

To ensure that the r.h.s. of the equation is always positive and to facilitate an analytic solution, we impose the relation

$$-\frac{\gamma}{|\gamma|} = r^2 \left(4A^2 + A(A+4B) \right). \tag{6.4}$$

Eq. (6.3) then yields,

$$\left(\hat{\Phi}'(\eta) \right)^2 = r^2 \mathrm{sech}^2(r\eta) \left(\frac{2A + B\mathrm{sech}^2(r\eta)}{A + B\mathrm{sech}^2(r\eta)} \right)^2, \tag{6.5}$$

and taking the square root of both sides and integrating yields the scalar field profile,

$$\hat{\Phi}(\eta) = \pm \left[\arctan\sinh(r\eta) + \sqrt{\frac{A}{A+B}} \arctan\left(\sqrt{\frac{A}{A+B}} \sinh(r\eta) \right) \right]. \tag{6.6}$$

The dimensionless potential $\hat{V}$ can easily be written as a function of η,

$$\hat{V}(\eta) = \left(\frac{r}{(A + B\mathrm{sech}^2(r\eta))} \right)^2 \times \Big[-2A^2 - 2A(3A+2B)\mathrm{sech}^2(r\eta) - \tag{6.7}$$

$$2B(A+B)\mathrm{sech}^4(r\eta) + \frac{5}{2} B^2 \mathrm{sech}^6(r\eta) \Big],$$

however $\hat{\Phi}(\eta)$ cannot easily be inverted analytically, so $\hat{V}$ can generally not be written analytically in terms of standard functions of $\hat{\Phi}$ (although $\hat{\Phi}$ is $1:1$ and therefore always invertible, so $\hat{V}$ can always be expressed numerically as a well-defined function of $\hat{\Phi}$).

AdS_4 (dS_4) brane cosmology corresponds to the case where $\gamma < 0$ ($\gamma > 0$), and by eq. (6.4), this is equivalent to requiring,

$$5A^2 + 4AB > 0, \ \text{AdS case}, \tag{6.8}$$

$$5A^2 + 4AB < 0, \ \text{dS case}. \tag{6.9}$$

In the AdS case, eq. (6.8) can obviously be easily satisfied provided $A, B > 0$. In this case $\sqrt{A/(A+B)}$ is real, and eq. (6.6) yields a kink-like profile (figure 5). This profile can be numerically inverted to give a plot of the generating potential $\hat{V}(\hat{\Phi})$ as a function of the scalar field $\hat{\Phi}$ (figure 6). As noted previously, the case $0 < A < B$ yields a metric qualitatively similar to that of [36], which is expected to localise gravity [22].

In the special case where $B = 0$, eq. (6.6) becomes,

$$\hat{\Phi}(\eta) = \pm 2 \arctan\sinh(r\eta). \tag{6.10}$$

Then the potential $V(\Phi(y))$ can be obtained analytically from eq. (6.8), by noting that

$$\mathrm{sech}^2(r\eta) = \frac{1}{1 + \sinh^2(r\eta)} = \cos^2\left(\frac{\hat{\Phi}}{2} \right), \tag{6.11}$$

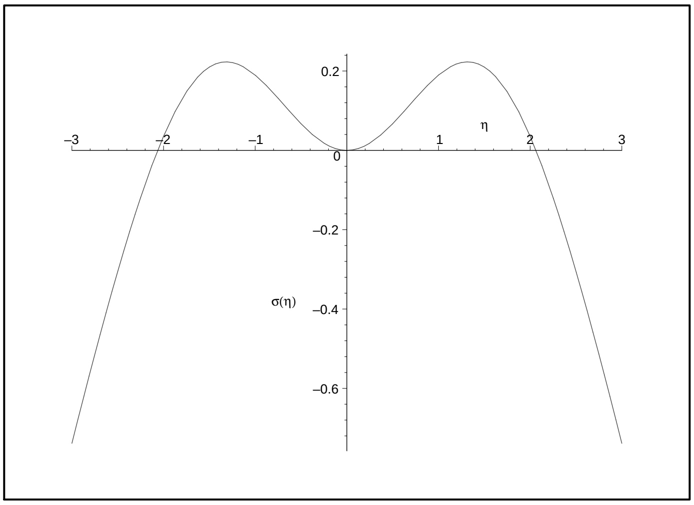

Figure 4: Warp factor exponent $\sigma(\eta)$ for a domain wall with AdS_4 brane cosmology; $A = 0.2, B = 0.8$, see eq. (6.1).

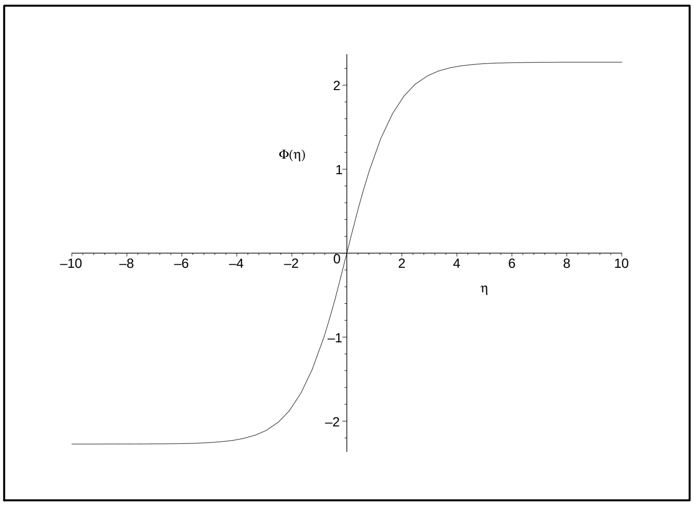

Figure 5: Dimensionless scalar field profile $\hat{\Phi}(\eta)$ for a domain wall with AdS_4 brane cosmology; $A = 0.2, B = 0.8$, see eqs. (6.1)-(6.6).

and so in this special case, the dimensionless potential takes the form,

$$\hat{V}(\eta) = -2r^2 \left[1 + 3\cos^2\left(\frac{\hat{\Phi}}{2}\right)\right]. \tag{6.12}$$

In fact, it is possible to obtain a slightly more general solution in this case: if we do not require eq. (6.4) to hold, then the scalar field configuration for $B = 0, \gamma < 0$ becomes,

$$\hat{\Phi}(\eta) = \frac{\pm\sqrt{1 - A^2 r^2}}{Ar} \operatorname{arctan} \sinh(r\eta), \tag{6.13}$$

and eq. (2.22) yields,

$$\hat{V}(\hat{\Phi}) = -2r^2 \left[1 - \frac{3}{4}\left(1 - \frac{1}{A^2 r^2}\right) \cos^2\left(\frac{AR}{\sqrt{1 - A^2 r^2}}\hat{\Phi}\right)\right], \tag{6.14}$$

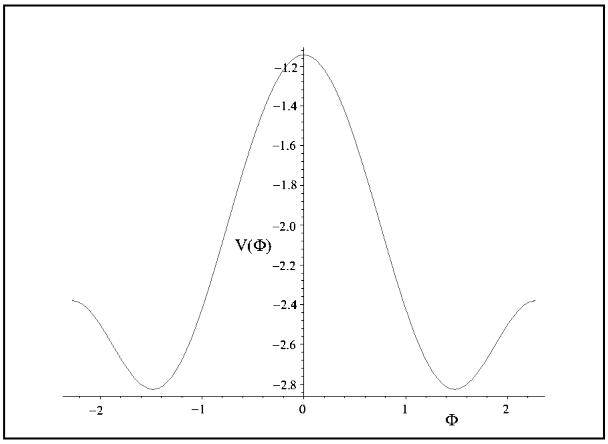

Figure 6: $\hat{V}(\hat{\Phi})$ vs $\hat{\Phi}$ for a domain wall with AdS_4 brane cosmology; $A = 0.2, B = 0.8$, see eqs. (6.1)-(6.6) .

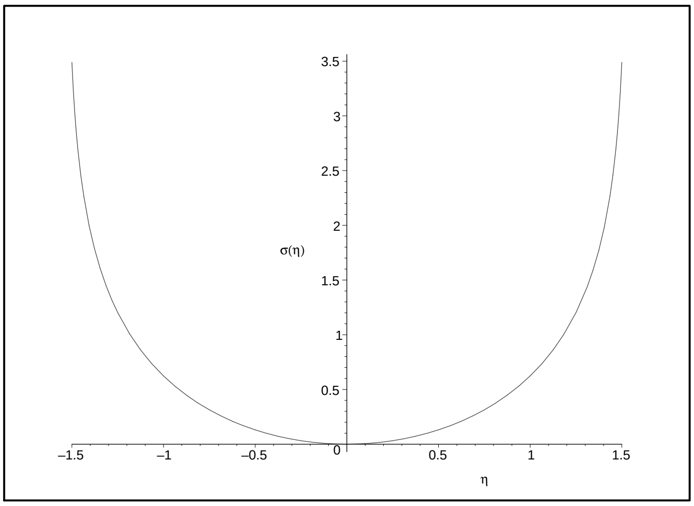

Figure 7: Warp factor exponent $\sigma(\eta)$ for a brane with dS_4 cosmology lying between two naked singularities; $A = -1, B = 2$, see eq. (6.1).

which reduces to eq. (6.12) in the case where $5A^2r^2 = 1$. Note that in these cases the asymptotes of the scalar field do not coincide with local minima of the potential.

Now in the dS case ($\gamma > 0$), it follows from eq. (6.9) that $4A(A + B) < 0$, and consequently $A/(A + B) < 0$. A sample profile of the warp factor exponent is shown in figure 7: as expected from the results of section 4, it diverges to ∞ a finite distance from the brane. (The warp factor exponent is obtained by taking $-\ln\omega(\eta)$; we plot the exponent rather than the warp factor itself to facilitate comparison with previous studies.)

Writing $i\alpha = \sqrt{A/(A + B)}$, for α real, the scalar field profile of eq. (6.6) becomes,

$$\hat{\Phi}(\eta) = \pm\left[\arctan\sinh(r\eta) - \alpha\,\mathrm{arctanh}\left(\alpha\sinh(r\eta)\right)\right]. \tag{6.15}$$

Let us assume a + sign for the purpose of this analysis: there are no significant differences between the kink and antikink solutions. A sample profile is given in figure 8.

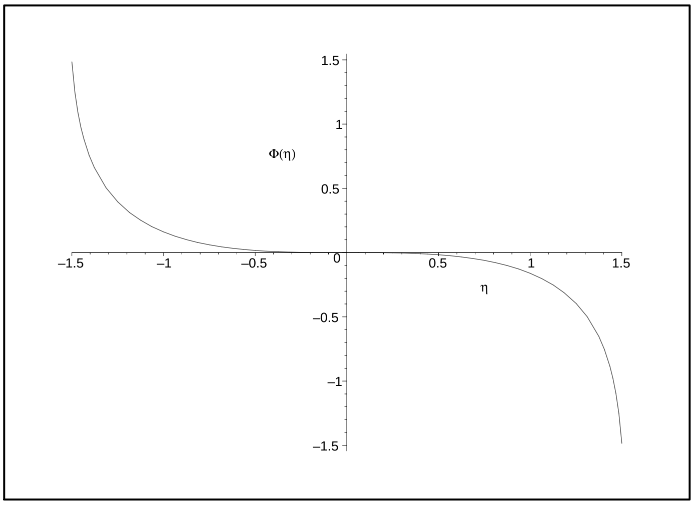

Figure 8: Dimensionless scalar field profile $\hat{\Phi}(\eta)$ supporting a brane with dS_4 cosmology; $A = -1, B = 2$, see eqs. (6.1), (6.15).

As is clear from figure 8, the second term diverges to $\pm\infty$ where $\alpha \sinh(r\eta) = \pm 1$: it is trivial to check that the points of divergence correspond precisely with the zeroes of the metric. Close to these singular points, which we shall denote $\pm\eta_0$, $\hat{\Phi}(\eta)$ can be approximated by,

$$\hat{\Phi}(-\eta_0 + \Delta\eta) = -\arctan\left(\frac{1}{\alpha}\right) - \frac{\alpha}{2}\ln\left(\frac{1}{2}r\alpha\sqrt{1+\alpha^2}\Delta\eta\right) + O(\Delta\eta), \tag{6.16}$$

$$\hat{\Phi}(\eta_0 - \Delta\eta) = \arctan\left(\frac{1}{\alpha}\right) + \frac{\alpha}{2}\ln\left(\frac{1}{2}r\alpha\sqrt{1+\alpha^2}\Delta\eta\right) + O(\Delta\eta). \tag{6.17}$$

Inverting these expressions allows the potential $\hat{V}$ to be written analytically as a function of $\hat{\Phi}$, close to the singularities. We find that the leading order term in $\hat{V}(\hat{\Phi})$ is given by,

$$\hat{V}(\hat{\Phi}) \sim -\frac{3}{32}\alpha^4\left(1+\alpha^2\right)e^{-\frac{4}{\alpha}\arctan\left(\frac{1}{\alpha}\right)}e^{\frac{4}{\alpha}|\hat{\Phi}|}, \tag{6.18}$$

so in this case $\hat{V}(\hat{\Phi})$ is unbounded below, and the scalar field diverges a finite distance from the brane. The potential can also be plotted numerically against the scalar field over the range between the singularities (figure 9), but there are no other features of real interest.

The 5D curvature scalar is given by eq. (4.7), and close to the singularities the leading order term is of the form,

$$R^{(5)}(\eta) \sim |\gamma|\alpha^2\frac{1}{(\Delta\eta)^2}. \tag{6.19}$$

Thus in this case the zeroes in the metric represent curvature singularities, and indicate divergences in the scalar field energy-momentum tensor, rather than merely being horizons.

7. Fermion trapping by warped metrics

The trapping of fermions on a scalar field domain wall was first described in the 1980s [1, 32]. More recently, fermion trapping in a scalar field domain wall with a five-dimensional warped

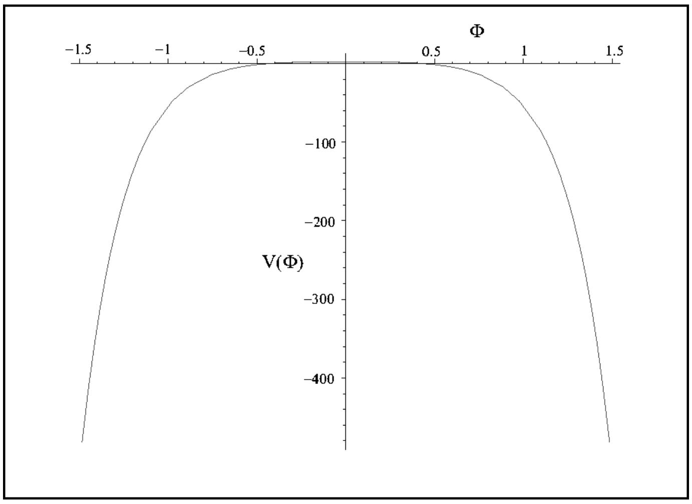

Figure 9: $\hat{V}(\hat{\Phi})$ vs $\hat{\Phi}$ for a potential supporting a brane with dS_4 cosmology; $A = -1, B = 2$, see eq. (6.1), (6.15).

metric has been discussed by a number of authors [14, 17, 18, 30, 33], but to our knowledge only in the case of a time-independent metric.

To write down the Dirac Lagrangian in curved spacetime, we employ the vielbeins V_M^A and inverse vielbeins V_A^M defined by [37, 39]:

$$g_{MN}(x) = V_M^A(x) V_N^B(x) \eta_{AB}. \tag{7.1}$$

Here η_{AB} is the Minkowski space metric, and $A, B, C, \ldots$ indicate coordinates in (locally defined) 5D Minkowski space, with $M, N, \ldots$ indicating coordinates in the curved space described by the metric g_{MN}.

The Dirac Lagrangian can then be written out as,

$$\mathcal{L}_\Psi \equiv \bar{\Psi} \Gamma^A \mathcal{D}_A \Psi - g_F \bar{\Psi} F(\Phi) \Psi, \tag{7.2}$$

where the Γ's are the Minkowski space Dirac matrices, F is some odd function of Φ describing the coupling between the Dirac field and the scalar field, and $\mathcal{D}_A$ is the covariant derivative with spin connection defined by,

$$\mathcal{D}_A \equiv V_A^M \left(\frac{\partial}{\partial x^M} + \Sigma_M(x) \right), \tag{7.3}$$

$$\Sigma_M(x) \equiv \frac{1}{2} \sigma^{BC} V_B^N V_{CN;M}, \tag{7.4}$$

$$V_{CN;M} = \frac{\partial V_{CN}}{\partial x^M} - \Gamma^R_{NM} V_{CR}, \tag{7.5}$$

$$\Gamma^R_{NM} = \frac{1}{2} g^{SR} \left[\frac{\partial g_{MN}}{\partial x^N} + \frac{\partial g_{NS}}{\partial x^M} - \frac{\partial g_{MN}}{\partial x^S} \right]. \tag{7.6}$$

Here σ^{AB} describes how the field transforms under infinitesimal Lorentz transformations, i.e. the spin of the field. For a spin-1/2 field, $\sigma^{AB} = (1/4)[\Gamma^A, \Gamma^B]$.

We may choose the Dirac matrices in five-dimensional Minkowski space, Γ^A, to be:

$$\Gamma^\alpha = \gamma^\alpha, \qquad \Gamma^4 = \gamma_5. \tag{7.7}$$

Here the γ matrices are simply the 4-dimensional Dirac matrices, which obey the four-dimensional Clifford algebra $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ with our chosen metric signature $(-+++)$.

It is easily verified that these Γ^A matrices obey the five-dimensional Clifford algebra, $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$, and it follows that for a Dirac field, if $A \neq B \Rightarrow \sigma^{AB} = (1/2)\Gamma^A\Gamma^B$ (for $A = B$, of course, $\sigma^{AB} = 0$). Note also that raising or lowering the indices on the Γ's is trivial: $\Gamma^0 = -\Gamma_0$, $\Gamma^A = \Gamma_A$ for $A \neq 0$.

As previously, we consider a 5D metric of the form given in eq. (2.1). For now, we shall employ the $\sigma(y)$ parameterisation, giving the key equations rewritten in terms of $\omega(y)$ at the end of this section. Let us choose a set of 4D vielbeins v_μ^α satisfying,

$$g^{(4)}_{\mu\nu} = v_\mu^\alpha v_\nu^\beta \eta_{\alpha\beta}. \tag{7.8}$$

Then the 5D vielbeins and inverse vielbeins may be chosen as,

$$V_M^A = \begin{pmatrix} e^{-\sigma(y)} v_\mu^\alpha & 0 \\ 0 & 1 \end{pmatrix}, \qquad V_A^M = \begin{pmatrix} e^{\sigma(y)} v_\alpha^\mu & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.9}$$

With this choice of vielbeins, the five-dimensional spin connection Σ_M becomes:

$$\Sigma_M = \begin{pmatrix} \Sigma_\mu^{(4)} + \frac{1}{2}\sigma'(y) e^{-\sigma(y)} \Gamma^4 \Gamma^\alpha v_{\alpha\mu} \\ 0 \end{pmatrix}, \tag{7.10}$$

where $\Sigma_\mu^{(4)}$ is the spin connection for fermion fields in the four-dimensional spacetime described by the metric $g^{(4)}_{\mu\nu}$.

Consequently, the covariant derivative with spin connection becomes

$$\mathcal{D}_\alpha = e^{\sigma(y)} v_\alpha^\mu \left(\frac{\partial}{\partial x^\mu} + \Sigma_\mu^{(4)} + \frac{1}{2}\sigma'(y) e^{-\sigma(y)} \Gamma^4 \Gamma^\beta v_{\beta\mu} \right), \qquad \mathcal{D}_4 = \frac{\partial}{\partial y}, \tag{7.11}$$

and the sum $\Gamma^A \mathcal{D}_A$ can be written,

$$\Gamma^A \mathcal{D}_A = e^{\sigma(y)} \left[\gamma^\alpha v_\alpha^\mu \left(\frac{\partial}{\partial x^\mu} + \Sigma_\mu^{(4)} \right) \right] + \gamma_5 \left(\frac{\partial}{\partial y} - 2\sigma'(y) \right). \tag{7.12}$$

The five-dimensional Dirac equation for a fermion field coupled to gravity and the background scalar field Φ is simply,

$$\left(\Gamma^A \mathcal{D}_A - g_F F(\Phi) \right) \Psi = 0. \tag{7.13}$$

It is well known [14, 17, 30, 33] that the confinement of fermion modes to the brane depends on their chirality. Let us therefore write the Dirac spinor Ψ in the form

$$\Psi(x) = \mathcal{U}_L(y) \psi_L(t, x^i) + \mathcal{U}_R(y) \psi_R(t, x^i), \tag{7.14}$$

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where ψ_L and ψ_R are the left-handed and right-handed components of a four-dimensional Dirac field, and hence $\gamma_5\psi_L = -\psi_L$, $\gamma_5\psi_R = \psi_R$. Inserting this ansatz and eq. (7.12) into the Dirac equation yields,

$$\begin{aligned}0 &= \left(\Gamma^A\mathcal{D}_A - g_F F(\Phi)\right)\Psi \\ &= \mathcal{U}_L(y)e^{\sigma(y)}\left[\gamma^\alpha v^\mu_\alpha\left(\frac{\partial}{\partial x^\mu} + \Sigma^{(4)}_\mu\right)\right]\psi_L(t,x^i) - \psi_L(t,x^i)\left(\frac{\mathrm{d}}{\mathrm{d}y} - 2\sigma'(y)\right)\mathcal{U}_L(y) + \\ &\quad +\mathcal{U}_R(y)e^{\sigma(y)}\left[\gamma^\alpha v^\mu_\alpha\left(\frac{\partial}{\partial x^\mu} + \Sigma^{(4)}_\mu\right)\right]\psi_R(t,x^i) + \psi_R(t,x^i)\left(\frac{\mathrm{d}}{\mathrm{d}y} - 2\sigma'(y)\right)\mathcal{U}_R(y) - \\ &\quad -g_F F(\Phi)\mathcal{U}_L(y)\psi_L(t,x^i) - g_F F(\Phi)\mathcal{U}_R(y)\psi_R(t,x^i). \end{aligned} \tag{7.15}$$

Now suppose we require that the four-dimensional spinors ψ_L, ψ_R satisfy the Dirac equation for fermions in the four-dimensional spacetime described by the metric $g^{(4)}_{\mu\nu}$,

$$\gamma^\alpha v^\mu_\alpha\left(\partial_\mu + \Sigma^{(4)}_\mu\right)\psi_L = m\psi_R, \tag{7.16}$$

$$\gamma^\alpha v^\mu_\alpha\left(\partial_\mu + \Sigma^{(4)}_\mu\right)\psi_R = m\psi_L. \tag{7.17}$$

Then the five-dimensional Dirac equation becomes,

$$\begin{aligned}0 = \psi_R(t,x^i)&\left[me^{\sigma(y)}\mathcal{U}_L(y) + \left(\frac{\mathrm{d}}{\mathrm{d}y} - 2\sigma'(y)\right)\mathcal{U}_R(y) - g_F F(\Phi)\mathcal{U}_R(y)\right] + \\ +\psi_L(t,x^i)&\left[me^{\sigma(y)}\mathcal{U}_R(y) - \left(\frac{\mathrm{d}}{\mathrm{d}y} - 2\sigma'(y)\right)\mathcal{U}_L(y) - g_F F(\Phi)\mathcal{U}_L(y)\right], \end{aligned} \tag{7.18}$$

and equating the coefficients of ψ_R and ψ_L separately to zero, we obtain a pair of coupled first-order differential equations in the bulk coordinate,

$$me^{\sigma(y)}\mathcal{U}_L(y) + \left(\frac{\mathrm{d}}{\mathrm{d}y} - 2\sigma'(y)\right)\mathcal{U}_R(y) - g_F F(\Phi)\mathcal{U}_R(y) = 0\,, \tag{7.19}$$

$$me^{\sigma(y)}\mathcal{U}_R(y) - \left(\frac{\mathrm{d}}{\mathrm{d}y} - 2\sigma'(y)\right)\mathcal{U}_L(y) - g_F F(\Phi)\mathcal{U}_L(y) = 0\,. \tag{7.20}$$

Note that these equations are identical to those for a static domain wall (see for example [14]): the generalisation to any 4D metric $g^{(4)}_{\mu\nu}$ only modifies the warp factor $\sigma(y)$ and the scalar field Φ.

Let us now define the dimensionless rescaled mass and coupling constant for the case $\gamma \neq 0$, in the case where $F(\Phi) = \Phi$,

$$\mu = m\sqrt{\frac{3}{|\gamma|}}, \qquad h_F = \frac{3g_F}{\kappa\sqrt{|\gamma|}}. \tag{7.21}$$

Then eq. (7.19)-(7.20) can be rewritten in terms of dimensionless quantities,

$$\mu e^{\sigma(\eta)}\mathcal{U}_L(\eta) + \left(\frac{\mathrm{d}}{\mathrm{d}\eta} - 2\sigma'(\eta)\right)\mathcal{U}_R(\eta) - h_F\hat{\Phi}(\eta)\mathcal{U}_R(\eta) = 0, \tag{7.22}$$

$$\mu e^{\sigma(\eta)}\mathcal{U}_R(\eta) - \left(\frac{\mathrm{d}}{\mathrm{d}\eta} - 2\sigma'(\eta)\right)\mathcal{U}_L(\eta) - h_F\hat{\Phi}(\eta)\mathcal{U}_L(\eta) = 0. \tag{7.23}$$

As previously, in configurations containing bulk singularities we may wish to rewrite the metric in terms of the warp factor ω: the only reason for casting our equations in terms of the warp factor exponent is for easy comparison with the existing literature. In this case the mode equations can be derived as,

$$\frac{m}{\omega(y)}\mathcal{U}_L(y) + \left(\frac{\mathrm{d}}{\mathrm{d}y} + 2\frac{\omega'(y)}{\omega(y)}\right)\mathcal{U}_R(y) - g_F F(\Phi)\mathcal{U}_R(y) = 0\,, \tag{7.24}$$

$$\frac{m}{\omega(y)}\mathcal{U}_R(y) - \left(\frac{\mathrm{d}}{\mathrm{d}y} + 2\frac{\omega'(y)}{\omega(y)}\right)\mathcal{U}_L(y) - g_F F(\Phi)\mathcal{U}_L(y) = 0\,, \tag{7.25}$$

or in terms of dimensionless quantities for the $F(\Phi) = \Phi$ case,

$$\frac{\mu}{\omega(\eta)}\mathcal{U}_L(\eta) + \left(\frac{\mathrm{d}}{\mathrm{d}\eta} + 2\frac{\omega'(\eta)}{\omega(\eta)}\right)\mathcal{U}_R(\eta) - h_F\hat{\Phi}(\eta)\mathcal{U}_R(\eta) = 0, \tag{7.26}$$

$$\frac{\mu}{\omega(\eta)}\mathcal{U}_R(\eta) - \left(\frac{\mathrm{d}}{\mathrm{d}\eta} + 2\frac{\omega'(\eta)}{\omega(\eta)}\right)\mathcal{U}_L(\eta) - h_F\hat{\Phi}(\eta)\mathcal{U}_L(\eta) = 0. \tag{7.27}$$

8. Confinement of the fermion zero mode on AdS$_4$ and dS$_4$ branes

In the case where $m = 0$, the equations for $\mathcal{U}_L$ and $\mathcal{U}_R$ simplify by decoupling,

$$\frac{\mathrm{d}\mathcal{U}_L(y)}{\mathrm{d}y} = \mathcal{U}_L(y)\left(2\sigma'(y) - g_F F(\Phi(y))\right), \tag{8.1}$$

$$\frac{\mathrm{d}\mathcal{U}_R(y)}{\mathrm{d}y} = \mathcal{U}_R(y)\left(2\sigma'(y) + g_F F(\Phi(y))\right), \tag{8.2}$$

allowing us to analytically study the confinement of the fermion zero mode. We are primarily interested in the behaviour of the fermion field within the region close to the brane and, in the case of dS brane cosmology, bounded by the singularities. We shall thus use the σ parameterisation, and demonstrate that the presence of even the coordinate singularities discussed in section 5 leads to the failure of the usual confinement mechanism.

The above first-order linear differential equations can easily be solved for the bulk coefficient functions,

$$\mathcal{U}_L(y) = A_L e^{2\sigma(y)} e^{-g_F \int F(\Phi(y))\mathrm{d}y}, \qquad \mathcal{U}_R(y) = A_R e^{2\sigma(y)} e^{g_F \int F(\Phi(y))\mathrm{d}y}. \tag{8.3}$$

If we substitute these expressions back into the kinetic-energy term of the Dirac Lagrangian, we obtain,

$$\begin{aligned}\bar{\Psi}\Gamma^A\mathcal{D}_A\Psi \,=\, & A_L^2 e^{5\sigma(y)} e^{-2g_F\int F(\Phi(y))\mathrm{d}y}\bar{\psi}_L(t,x^i)\left[\gamma^\alpha v^\mu_\alpha\left(\frac{\partial}{\partial x^\mu} + \Sigma^{(4)}_\mu\right)\right]\psi_L(t,x^i) + \\ & + A_R^2 e^{5\sigma(y)} e^{2g_F\int F(\Phi(y))\mathrm{d}y}\bar{\psi}_R(t,x^i)\left[\gamma^\alpha v^\mu_\alpha\left(\frac{\partial}{\partial x^\mu} + \Sigma^{(4)}_\mu\right)\right]\psi_R(t,x^i) + \\ & + g_F F(\Phi) A_L A_R e^{4\sigma(y)}\left(\bar{\psi}_R(t,x^i)\psi_L(t,x^i) + \bar{\psi}_L(t,x^i)\psi_R(t,x^i)\right).\end{aligned} \tag{8.4}$$

The action for the fermion field is simply,

$$S_\Psi = \int \mathrm{d}^5x\sqrt{-g(x)}\mathcal{L}_\Psi(x). \tag{8.5}$$

If $\hat{g} = Det(g^{(4)}_{\mu\nu})$, then $g = e^{-8\sigma(y)}\hat{g}$, and $\sqrt{-g} = e^{-4\sigma(y)}\sqrt{-\hat{g}}$. Noting that the term in $F(\Phi(y))$ is odd for a kink solution, we need only consider the first two terms in the action, because the third integrates to zero. These terms factorise into the usual 4D action over the brane coordinates,

$$\int_{-\infty}^{\infty} \mathrm{d}^4x \sqrt{-\hat{g}} \left[\bar{\psi}_{L/R}(t, x^i) \left(\gamma^\alpha v^\mu_\alpha \left(\frac{\partial}{\partial x^\mu} + \Sigma^{(4)}_\mu \right) \right) \psi_{L/R}(t, x^i) \right], \tag{8.6}$$

multiplied by integrals over the bulk coordinates given by

$$A_L^2 \int_{-\infty}^{\infty} \mathrm{d}y\; e^{\sigma(y)} e^{-2g_F \int F(\Phi(y))\mathrm{d}y}, \qquad A_R^2 \int_{-\infty}^{\infty} \mathrm{d}y\; e^{\sigma(y)} e^{2g_F \int F(\Phi(y))\mathrm{d}y}, \tag{8.7}$$

for left-handed and right-handed fermion fields respectively.

For the case where $F(\Phi) = \Phi$ and $\gamma \neq 0$, the normalisation integrals can be written in terms of the dimensionless quantities,

$$A_L^2 \sqrt{\frac{3}{|\gamma|}} \int_{-\infty}^{\infty} \mathrm{d}\eta\; e^{\sigma(\eta)} e^{-2h_F \int \hat{\Phi}(\eta)\mathrm{d}\eta}, \qquad A_R^2 \sqrt{\frac{3}{|\gamma|}} \int_{-\infty}^{\infty} \mathrm{d}\eta\; e^{\sigma(\eta)} e^{2h_F \int \hat{\Phi}(\eta)\mathrm{d}\eta}. \tag{8.8}$$

It is now clear that in general, zeroes in the metric (points where $\sigma(\eta)$ diverges to ∞) correspond to singularities in the integrand, and thus the kinetic energy part of the action may be expected to be non-normalisable in cases where the metric contains zeroes. In particular, this will be the case for the new solution presented in eqs. (5.8)-(5.14). This obviously represents a challenge for that kind of model: although it has the nice properties that the bulk inverse-metric singularity is a coordinate not a curvature singularity and the scalar field configuration is smooth, it has the drawback that the usual kink-based fermion zero-mode localisation mechanism does not generalise to that dS_4 case. Dealing with this challenge is beyond the scope of this paper, but two logical approaches immediately suggest themselves: One could look to introduce new physics, beyond the scalar kink, to localise fermions. Alternatively, we know that our universe is only approximately de Sitter, so it would be interesting to explore fermion localisation via the kink mechanism in an effective domain wall FRW cosmology that has the transitions from radiation to matter to vacuum energy domination of regular cosmology.

It seems possible that if the kink Φ also diverges to $\pm\infty$ at the metric zeroes, then one of the chiral fermion zero modes might be normalisable over the bulk. However, in the dS_4 solution we have described with curvature singularities at the metric zeroes, this cancellation does not occur. Eqs. (6.16)-(6.17) show that close to the singularities, the leading order term of $\hat{\Phi}(\eta)$ is of the form $\ln(q\Delta\eta)$, so the integral function $\int \hat{\Phi}(\eta)\,\mathrm{d}\eta$ is bounded close to the singularities, while $e^{\sigma(\eta)}$ diverges as $1/\Delta\eta$.

If the warp factor exponent $\sigma(y)$ is smooth and nonsingular, the fermion zero modes may or may not be normalisable. Suppose $\sigma(y) \sim c|y|$ as $y \to \pm\infty$, for some $c < 0$. Then one of the two chiral fermion zero modes will always be confined: the chirality of this mode will depend on the scalar field profile. The other mode may or may not be confined, depending on the asymptotic behaviour of $\int \mathrm{d}y\, F(\Phi(y))$ relative to $\sigma(y)$. This is the case when the metric diverges to infinity asymptotically, and is relevant for the cases considered in section 6, with AdS_4 cosmology on the brane. This behaviour has also been studied by Koley and Kar in the context of a "ghost" scalar field with negative energy in the bulk in [31], and by Bajc and Gabadadze for fermions localised on a non-fine-tuned RS brane by a scalar field in [33].

If conversely the warp factor exponent behaves as $\sigma(y) \sim c|y|$ for $y \to \pm\infty$, with $c > 0$, then one of the two modes will certainly be non-normalisable, while the other may be normalisable depending on the behaviour of the integral $\int \mathrm{d}y\, F(\Phi(y))$. This is the case for fine-tuned static brane solutions [14, 32, 33], where the metric elements approach zero for large $|y|$. This type of warp factor is employed to localise gravity on the brane in the Randall-Sundrum approach [4].

In particular, if $F(\Phi(y))$ has a kink or anti-kink profile, and $\lim_{y\to\infty} F(\Phi(y)) = -\lim_{y\to-\infty} F(\Phi(y)) = \lambda$, then far from the brane, $\int \mathrm{d}y F(\Phi(y)) \sim \lambda|y|$ (up to a constant of integration). Consequently, if $\sigma(y) \sim c|y|$ for $|y|$ large, then the normalisation integrands are of the form,

$$\exp\left((c + 2g_F\lambda)\,|y|\right), \qquad \exp\left((c - 2g_F\lambda)\,|y|\right), \tag{8.9}$$

for right- and left-handed fields respectively.

For the "locally localised gravity" AdS_4 solution outlined in section 6, these conditions hold for the simple coupling $F(\Phi) = \Phi$, with the parameters,

$$\lambda = \frac{\sqrt{3}}{\kappa}\frac{\pi}{2}\,(1+\alpha)\,, \qquad c = -r. \tag{8.10}$$

Thus the action is normalisable for right-handed fermion fields provided,

$$g_F < \frac{\kappa}{\sqrt{3}}\frac{r}{\pi\,(1+\alpha)}, \tag{8.11}$$

and for left-handed fermion fields if

$$g_F > -\frac{\kappa}{\sqrt{3}}\frac{r}{\pi\,(1+\alpha)}, \tag{8.12}$$

but the latter relation is always true for positive g_F.

In terms of the dimensionless quantities, we replace y with η and g_F with h_F in the discussion above, and the parameters become $c = -r$, $\lambda = \pi/2(1+\alpha)$. The normalisability conditions become,

$$h_F < \frac{r}{\pi\,(1+\alpha)} \tag{8.13}$$

for right-handed fermions, and

$$h_F > -\frac{r}{\pi\,(1+\alpha)}, \tag{8.14}$$

for left-handed fermions.

However, normalisability of the chiral fermion zero-modes does not necessarily imply that those modes are localised on the brane. For example, in some parameter regimes, the normalisation integrand has two peaks at the points where the warp factor turns around, and a local minimum (rather than a local maximum) on the brane, before falling off rapidly far from the brane. If the local minimum is sufficiently shallow and the peaks are sufficiently close to the brane, this behaviour may still constitute localisation to the brane.

The Lagrangian density always has a local extremum at the brane (i.e. its derivative with respect to the bulk coordinate is zero there): its second derivative with respect to the bulk coordinate determines whether the brane lies at a local minimum or maximum. The second derivative of the integrand of eq. (8.8) has the same sign as

$$\sigma''(\eta) \pm 2h_F\hat{\Phi}'(\eta), \tag{8.15}$$

where as previously, the $+$ sign applies to right-handed fermion fields and the $-$ sign corresponds to left-handed fields.

For the AdS_4 case outlined in section 6,

$$\sigma''(0) \pm 2h_F\hat{\Phi}'(0) = r^2\left(1-2\alpha^2\right) \pm 2h_F r\left(1+\alpha^2\right), \tag{8.16}$$

so the brane lies at a local maximum of the Lagrangian density for,

$$\pm h_F < \frac{r\left(2\alpha^2-1\right)}{2\left(\alpha^2+1\right)}. \tag{8.17}$$

Note that in the case where $0 < A < B$, $\alpha < 1/2$ and therefore the r.h.s. of this expression is always negative. Consequently, this condition cannot hold for right-handed fermion modes: this result is analogous to the usual behaviour of fermion zero modes for a Minkowski brane, where right-handed modes are always unconfined [14, 33]. Figure 10 demonstrates a sample profile of the integrand of eq. (8.8) for the case where the right-handed fermion modes are normalisable, but not confined to the brane. Left-handed fermion modes are always normalisable, and may (figure 11) or may not (figure 12) exhibit a peak at the brane itself.

9. Conclusion

We have presented solutions to the Einstein equations describing a scalar field coupled to five-dimensional gravity, with a warped five-dimensional metric and dS_4 and AdS_4 brane cosmology. In the dS_4 case, the metric necessarily contains zeroes which generally (but not inevitably) correspond to curvature singularities. It is possible to obtain a warped metric with dS_4 slices from a smooth potential and scalar field, and in this case the metric zeroes are simply horizons. However, there are stringent conditions on the metric in this case which make it difficult to write down an analytic solution. Moreover, the presence of zeroes in the five-dimensional warped metric, whether representing horizons or curvature singularities, generally leads to divergences in the normalisation integrals for both the left- and right-handed zero modes of the Dirac field. This issue does not arise in the case of an infinitely thin brane, where the matter fields are confined a priori to a 3+1-dimensional slice of the higher-dimensional space and do not extend into the bulk at all.

In the AdS_4 case, we have derived new analytic self-consistent solutions for the metric warp factor and scalar field kink, motivated by the principle that localisation of gravity

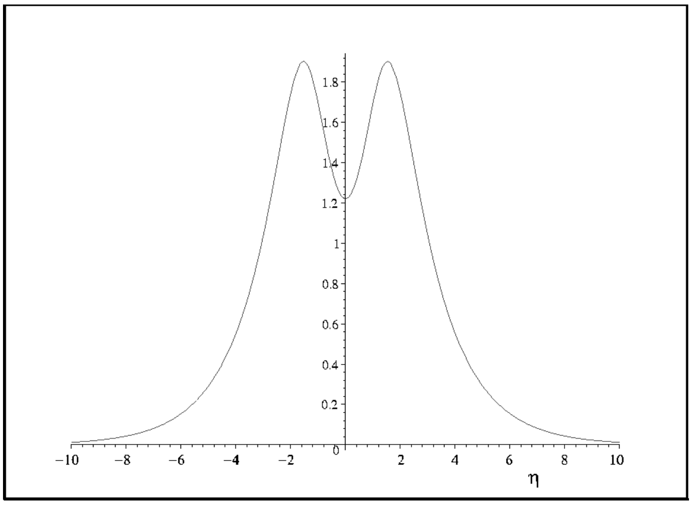

Figure 10: Bulk dependence of the kinetic energy term, eq. (8.8), for right-handed fermions on a domain wall with AdS_4 brane cosmology; $A = 0.2, B = 0.8, h_F = 0.1$; see eqs. (6.1)-(6.8) and eq. (7.21).

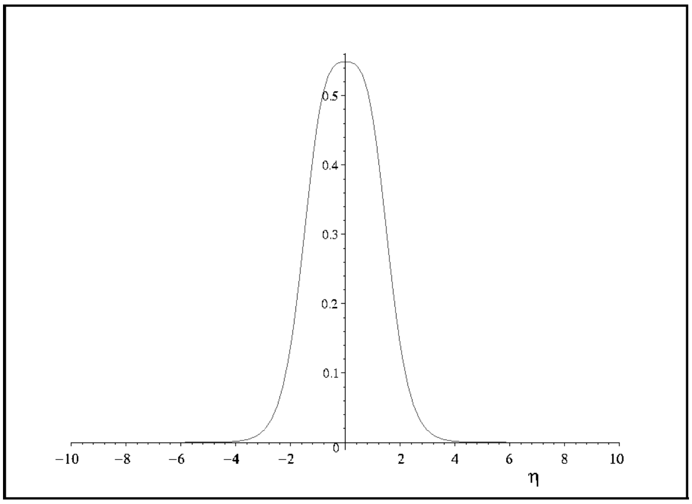

Figure 11: Bulk dependence of the kinetic energy term, eq. (8.8), for left-handed fermions on a domain wall with AdS_4 brane cosmology; $A = 0.2, B = 0.8, h_F = 0.3$; see eqs. (6.1)-(6.8) and eq. (7.21).

should depend only on the behaviour of the metric close to the brane [36]. Although for non-trivial brane cosmology the scalar field does not generally seem to asymptote to minima of the potential, these configurations (and their dS_4 counterparts) are expected to be both classically stable and to confine the four-dimensional graviton [22]. The ensuing warped metric admits both left- and right-handed normalisable fermionic zero modes (as in [31, 33]), although not all the normalisable modes are localised on the brane. At present we have only investigated massless chiral fermions, but it might also be interesting to investigate the spectrum of massive fermionic modes in these backgrounds, using eqs. (7.19)-(7.20).

Our present analysis only applies to a particular class of warped metrics, corresponding

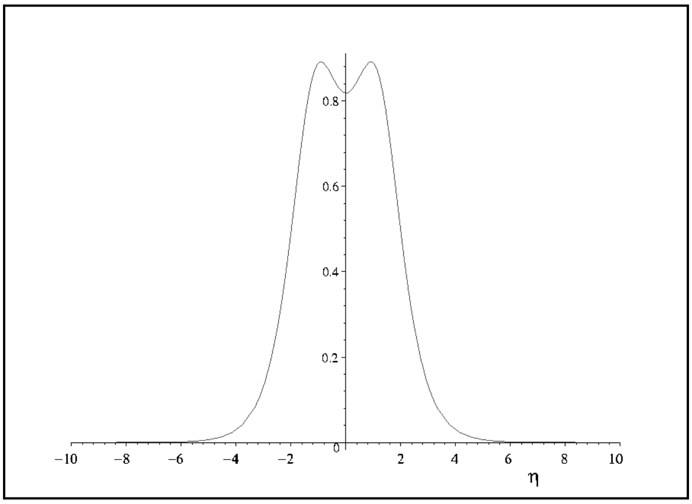

Figure 12: Bulk dependence of the kinetic energy term, eq. (8.8), for left-handed fermions on a domain wall with AdS_4 brane cosmology; $A = 0.2, B = 0.8, h_F = 0.1$; see eqs. (6.1)-(6.8) and eq. (7.21).

to four-dimensional metrics of constant curvature. Future studies dealing with the effects of matter on the brane will need to employ a more general metric ansatz, allowing metric elements that are non-separable functions of the brane and bulk coordinates.

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